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Cardy-Verlinde Formula and AdS Black Holes

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Abstract

In a recent paper hep-th/0008140 by E. Verlinde, an interesting formula has been put forward, which relates the entropy of a conformal formal field in arbitrary dimensions to its total energy and Casimir energy. This formula has been shown to hold for the conformal field theories that have AdS duals in the cases of AdS Schwarzschild black holes and AdS Kerr black holes. In this paper we further check this formula with various black holes with AdS asymptotics. For the hyperbolic AdS black holes, the Cardy-Verlinde formula is found to hold if we choose the "massless" black hole as the ground state, but in this case, the Casimir energy is negative. For the AdS Reissner-Nordström black holes in arbitrary dimensions and charged black holes in D=5, D=4, and D=7 maximally supersymmetric gauged supergravities, the Cardy-Verlinde formula holds as well, but a proper internal energy which corresponds to the mass of supersymmetric backgrounds must be subtracted from the total energy. It is failed to rewrite the entropy of corresponding conformal field theories in terms of the Cardy-Verlinde formula for the AdS black holes in the Lovelock gravity.

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1 Introduction

It is well-known that the entropy of a $(1+1)$-dimensional conformal field theory (CFT) can be given by the Cardy formula [1]

$$S = 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)}, \tag{1.1}$$

where c is the central charge, L_0 denotes the product ER of the total energy and radius of system, and the shift of $c/24$ is caused by the Casimir effect, which is a finite volume effect.

In a recent paper by Erik Verlinde [2], it has been proposed that the Cardy formula (1.1) can be generalized to the case in arbitrary dimensions. Consider a conformal field theory living in $(1+n)$-dimensional spacetime described by the metric

$$ds^2 = -dt^2 + R^2 d\Omega_n^2, \tag{1.2}$$

where R is the radius of a n-dimensional sphere. The entropy of the CFT can be given by the generalized Cardy formula (hereafter we refer to this as the Cardy-Verlinde formula),

$$S = \frac{2\pi R}{\sqrt{ab}}\sqrt{E_c(2E - E_c)}, \tag{1.3}$$

where E_c represents the Casimir energy, a and b are two positive coefficients which are independent of R and S. For strong coupled CFT's with AdS duals, the value of product ab is fixed to n^2 exactly. The above Cardy-Verlinde formula is then reduced to

$$S = \frac{2\pi R}{n}\sqrt{E_c(2E - E_c)}. \tag{1.4}$$

With given the total energy E and the radius R, the Cardy-Verlinde formula (1.4) gives the maximal entropy

$$S \leq S_{\rm max} = \frac{2\pi RE}{n}, \tag{1.5}$$

when $E_c = E$. This is just the Bekenstein entropy bound [3][1].

In the spirit of AdS/CFT correspondence [5, 6, 7], It was convincingly argued by Witten [8] that the thermodynamics of a certain CFT at high temperature can be identified with the thermodynamics of black holes in anti-de Sitter space (AdS). With this

[1] The Bekenstein entropy bound states that the ratio of the entropy S to the total energy E of a closed physical system with limited self-gravity, which fits in a sphere with radius R in three spatial dimensions, obeys $S \leq 2\pi RE$. In fact, the Bekenstein entropy bound is independent of the spatial dimension. For a derivation of the Bekenstein bound in arbitrary dimensions see [4].

correspondence, Verlinde checked the formula (1.4) using the thermodynamics of AdS Schwarzschild black holes in arbitrary dimensions and found it holds exactly [2]. Furthermore the Cardy-Verlinde formula has been checked more recently for the AdS Kerr black holes in [9], which corresponds to a CFT residing in a rotating Einstein universe. Once again, this formula has been found to hold exactly. Some of recent works related to the entropy bound and the Cardy-Verlinde formula are [10, 11, 12, 13, 14, 15].

No doubt, it is of great interest to prove the validity of the Cardy-Verlinde formula for all CFT's. Of course, it might be a quite difficult task. Having considered the absence of such a proof so far, it is worthwhile to do some further check for the Cardy-Verlinde formula in a larger extent than that in [2] and [9], in order to see to what extent this formula is valid. This is just the aim of this paper.

In this paper we choose some typical examples of black holes with AdS asymptotics to check the Cardy-Verlinde formula. In a $(n+2)$-dimensional AdS, except for the spherically symmetric AdS Schwarzschild black holes whose horizon is a n-dimensional sphere surface with positive constant curvature, There exist the so-called hyperbolic AdS black holes whose horizon is a negative constant curvature hypersurface. The thermodynamics of the latter is different from that of the former. It would be interesting to see if the Cardy-Verlinde formula holds in this case. This will be done in the next section.

The gauged supergravity can also be realized as a self-consistent truncation of superstring or M theory compactified on a compact manifold. The gauge group is the isometry group of the compact manifold. In the course of AdS/CFT correspondence, the gauge group is identified with the R-symmetry group of boundary CFT's. In this sense the thermodynamics of AdS charged black holes can be viewed as that of a certain CFT with a chemical potential. So we will check the Cardy-Verlinde formula in section 3 with AdS Reissner-Nordström black holes. There we will also discuss the charged black holes in D=5, D=4 and D=7 maximally supersymmetric gauged supergravities. These theories can be regarded as the self-consistent truncations of IIB supergravity on the S^5, 11-dimensional supergravity on the S^7 and S^4, respectively.

In supergravity theories, higher derivative curvature terms occur as the corrections of the massive string states and string loop corrections in superstring theories. In the AdS/CFT correspondence, these corrections correspond to those of large N expansion of boundary CFT's in the strong coupling limit. So it is also interesting to see if the Cardy-Verlinde formula still remains valid after including some of those corrections. However, it

is in general quite difficult to find exact nontrivial black hole solutions in higher derivative gravity theories, which are required for exactly checking the Cardy-Verlinde formula. In section 4 we will consider a special kind of Lovelock gravity theories, in which by choosing some special coefficients for each term in the action, a simple, but exact black hole solution can be found. Using this black hole solution, we examine the thermodynamics of corresponding CFT's. We summarize our results in section 5 with brief discussions.

2 Hyperbolic AdS black holes in arbitrary dimensions

In four dimensional spacetimes, it is believed generally that the horizon of a black hole must be a sphere S^2, up to diffeomorphisms. However, it can be violated if the theory includes a negative cosmological constant. It was already found that except for the sphere case, namely, the horizon is a positive constant curvature hypersurface, black holes are allowed to exist with horizon which are zero or negative constant curvature hypersurface. In higher dimensional ($D \geq 4$) spaces, it is true as well. For those so-called topological black hole solutions in arbitrary dimensions see [16].

For a ($D = n+2$)-dimensional spherically symmetric black hole in AdS spacetime, its thermodynamics corresponds to the one of a CFT living in $(n+1)$-dimensional spacetime with topology $R \times S^n$. This case has been already checked in [2]. For black holes with zero curvature horizon, its thermodynamics is conformal invariant and the Casimir energy vanishes. So the Cardy-Verlinde formula (1.4) is not applicable in this case. As a result, in this section we discuss the AdS black holes with negative constant curvature horizon. In this case, its thermodynamics corresponds to the one of a CFT residing in a spacetime $R \times \Sigma_g^n$, where Σ_g^n denotes a n-dimensional negative constant curvature space, which can be a closed hypersurface with arbitrary high genus under appropriate identification.

The metric for a hyperbolic AdS black hole in a $(n+2)$-dimensional spacetime can be written down as [16]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Sigma_n^2, \tag{2.1}$$

where

$$f(r) = -1 - \frac{\omega_n M}{r^{n-1}} + \frac{r^2}{l^2}, \quad \omega_n = \frac{16\pi G}{nVol(\Sigma_n)}, \tag{2.2}$$

$d\Sigma_n^2$ denotes the line element of a n-dimensional hypersurface with constant curvature

$-n(n+1)$, $Vol(\Sigma_n)$ stands for the volume of the hypersurface Σ_n, and G is the Newtonian gravity constant. This is a solution of Einstein equations with a negative cosmological constant $\Lambda = -n(n+1)/2l^2$.

The solution (2.1) has some peculiar properties in the sense: (1) When the integration constant $M = 0$, even though the solution is locally an anti-de Sitter space, it has a black hole horizon $r_+ = l$ with Hawking temperature $T_{\rm HK}$ and Bekenstein-Hawking entropy S,

$$T_{\rm HK} = \frac{1}{2\pi l}, \quad S = \frac{l^n Vol(\Sigma_n)}{4G}. \tag{2.3}$$

This is the so-called "massless" black hole; (2) When $M > 0$, the solution (2.1) has only a black hole horizon satisfying

$$M = \frac{r_+^{n-1}}{\omega_n}\left(\frac{r_+^2}{l^2} - 1\right), \tag{2.4}$$

which implies that $r_+ > l$. When $M < 0$, however, it can have two black hole horizons, which coincide as

$$M = M_{\rm ext} = -\left(\frac{2}{n+1}\right)\left(\frac{n-1}{n+1}\right)^{(n-1)/2}\frac{l^{n-1}}{\omega_n}. \tag{2.5}$$

In this case, the coincident horizon $r_+^2 = l^2(n-1)/(n+1)$, the Hawking temperature vanishes, the black hole becomes an extremal one. It is the peculiar property which causes the difficulty to choose an appropriate reference background in order to determine the mass of hyperbolic black holes [16, 17]. In other words, there are some debates about the ground state of the hyperbolic AdS black holes.

Let us first suppose the "massless" black hole (2.3) as the ground state of the hyperbolic AdS black holes (2.1). In this case, the constant M is the mass of black holes, the temperature and entropy of black holes are

$$\begin{aligned} T_{\rm HK} &= \frac{1}{4\pi r_+}\left(\frac{(n+1)r_+^2}{l^2} - (n-1)\right), \\ S &= \frac{r_+^n Vol(\Sigma_n)}{4G}. \end{aligned} \tag{2.6}$$

According to the prescription of AdS/CFT correspondence [6, 7], the boundary spacetime in which the boundary CFT resides can be determined using the bulk metric, up to a conformal factor. It is due to the conformal factor that one can arbitrary rescale the boundary metric as one wishes. In this paper, we rescale the boundary metric so that the

finite volume has a radius R (this implies that $T > 1/R$ is assumed for the temperature T of corresponding CFT's) [2]. That is, the boundary metric has the following form

$$ds_b^2 = \lim_{r\to\infty} \frac{R^2}{r^2} ds^2 = -\frac{R^2}{l^2}dt^2 + R^2 d\Sigma_n^2. \tag{2.7}$$

Thus the system has the finite volume $V = R^n Vol(\Sigma_n)$. The $(n+1)$-dimensional CFT corresponding to the hyperbolic black holes has the energy E, temperature T and entropy S in the metric (1.2),

$$\begin{aligned} E &= \frac{nlVol(\Sigma_n)r_+^{n-1}}{16\pi GR}\left(\frac{r_+^2}{l^2}-1\right), \\ T &= \frac{l}{4\pi R r_+}\left(\frac{(n+1)r_+^2}{l^2}-(n-1)\right), \\ S &= \frac{r_+^n Vol(\Sigma_n)}{4G}. \end{aligned} \tag{2.8}$$

Following [2], let us define the Casimir energy E_c as

$$E_c = n(E + pV - TS), \tag{2.9}$$

where p represents the pressure of CFT defined as $p = -\left(\frac{\partial E}{\partial V}\right)_S$. With the help of (2.8), we obtain

$$E_c = -2\frac{nlr_+^{n-1}Vol(\Sigma_n)}{16\pi GR}. \tag{2.10}$$

Furthermore we have the extensive energy

$$2E - E_c = 2\frac{nr_+^{n+1}Vol(\Sigma_n)}{16\pi GRl}. \tag{2.11}$$

With these quantities, we find that the entropy S of CFT in (2.8) can be rewritten as

$$S = \frac{2\pi R}{n}\sqrt{|E_c|(2E-E_c)}. \tag{2.12}$$

Comparing with the Cardy-Verlinde formula (1.4), we find that indeed the entropy of a CFT residing in a finite hyperbolic space can also be expressed in a form of the Cardy-Verlinde formula. But, we find the Casimir energy is negative in this case! This feature can be traced back to the peculiar properties of hyperbolic AdS black holes discussed above. In other words, it is related to the existence of the "massless" black holes and "negative mass" black holes.

[2] In [2] the radius is taken to be the horizon radius of black holes.

One may wonder if the difficulty of negative Casimir energy can be circumvented through choosing the extremal black hole (2.5) as the ground state of black holes. In that case, the thermodynamics of black holes is still given by (2.6), but the mass of black holes becomes $M - M_{\rm ext}$. After a simple repeat as the above, one has to be led to the conclusion that the entropy cannot be expressed in terms of the Cardy-Verlinde formula in this case.

3 Charged AdS black holes

3.1 AdS Reissner-Nordström black holes in arbitrary dimensions

The metric of a $(n+2)$-dimensional AdS Reissner-Nordström black hole is [18, 19]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_n^2, \tag{3.1}$$

where $d\Omega_n^2$ denotes the line element of a unit n-dimensional sphere and the function f is given by

$$f(r) = 1 - \frac{m}{r^{n-1}} + \frac{\tilde{q}^2}{r^{2n-2}} + \frac{r^2}{l^2}. \tag{3.2}$$

When $m/2 = |\tilde{q}|$, this solution is supersymmetric and the function f has the form

$$f(r) = \left(1 - \frac{m}{2r^{n-1}}\right)^2 + \frac{r^2}{l^2}. \tag{3.3}$$

Obviously, in this case, this solution does not represent a black hole and the singularity at $r = 0$ becomes naked.

For the convenience of discussions below, let us rewrite the solution (3.1) in terms of "isotropic" coordinates. Defining

$$m = \mu + 2q, \quad \tilde{q}^2 = q(\mu + q), \quad r^{n-1} \to r^{n-1} + q, \tag{3.4}$$

we can change (3.1) to the following form

$$ds^2 = H^{-2}f(r)dt^2 + H^{2/(n-1)}(f(r)^{-1}dr^2 + r^2 d\Omega_n^2), \tag{3.5}$$

where

$$f(r) = 1 - \frac{\mu}{r^{n-1}} + \frac{r^2}{l^2}H^{2n/(n-1)}, \quad H = 1 + \frac{q}{r^{n-1}}. \tag{3.6}$$

In this coordinates, the supersymmetric solution corresponds to the case when $\mu = 0$. The horizon r_+ of black hole is determined by the equation

$$\mu = r_+^{n-1} + \frac{\rho^{2n/(n-1)}}{l^2 r_+^{n-1}}, \quad \rho = r_+^{n-1} + q. \tag{3.7}$$

To find the thermodynamic quantities of black hole is straightforward. The mass M, entropy S and Hawking temperature $T_{\rm HK}$ are

$$\begin{aligned} M &= \frac{nVol(S^n)}{16\pi G}(\mu + 2q), \\ S &= \frac{Vol(S^n)}{4G}\rho^{n/(n-1)}, \\ T_{\rm HK} &= \frac{r_+^{n-1}}{4\pi\rho^{n/(n-1)}}\left((n-1) + \frac{2n\rho^{2n/(n-1)}}{l^2 r_+^{n-1}} - \frac{(n-1)\rho^{2n/(n-1)}}{l^2 r_+^{2n-2}}\right), \end{aligned} \tag{3.8}$$

respectively, where $Vol(S^n)$ stands for the volume of the unit n-dimensional sphere. The chemical potential ϕ associated with the physical electric charge $\tilde{q}$ is

$$\phi = \frac{nVol(S^n)}{16\pi G}\frac{2\tilde{q}}{\rho}. \tag{3.9}$$

As expected, these thermodynamic quantities satisfy the first law of black hole thermodynamics

$$dM = T_{\rm HK}dS + \phi d\tilde{q}. \tag{3.10}$$

Rescaling the boundary metric so that the n-dimensional sphere has the radius R and the volume $V = R^n Vol(S^n)$, in the spirit of AdS/CFT correspondence, we have the energy, temperature, and chemical potential of the corresponding CFT in the metric (1.2),

$$E = \frac{l}{R}M, \quad T = \frac{l}{R}T_{\rm HK}, \quad \Phi = \frac{l}{R}\phi, \tag{3.11}$$

respectively. The entropy and electric charge of CFT are still given by S and $\tilde{q}$. From (3.8), we can see that the energy of CFT can be separated to two parts: one of them is proportional to q, which is the contribution of supersymmetric background; the other is proportional to μ, which corresponds to the contribution of thermodynamic excitations. Let us define

$$E_q = \frac{nlVol(S^n)}{16\pi GR} \cdot 2q. \tag{3.12}$$

Reasonably, we can view the proper internal energy E_q as the zero temperature energy of CFT, which makes the contribution to the free energy, but not to the entropy. Following [2], we define the Casimir energy in this case as

$$E_c = n(E + pV - TS - \Phi\tilde{q}), \tag{3.13}$$

where the pressure p is defined as $p = -\left(\frac{\partial E}{\partial V}\right)_{S,\tilde{q}}$. Here it is worth stressing that when the electric charge vanishes, we have the pressure $p = E/nV$ because we are considering CFT's. When the electric charge does not vanish, however, the proper internal energy (zero temperature energy) E_q does not make its contribution to the pressure. Therefore the pressure should have the following form

$$p = \frac{E - E_q}{nV}. \tag{3.14}$$

With this relation, substituting those quantities in (3.11) into (3.13) yields

$$E_c = 2\frac{nlVol(S^n)r_+^{n-1}}{16\pi GR}. \tag{3.15}$$

Using this Casimir energy, it is easy to find that the entropy of corresponding CFT's for the AdS Reissner-Nordström black holes can be expressed as

$$S = \frac{2\pi R}{n}\sqrt{E_c(2(E - E_q) - E_c)}, \tag{3.16}$$

where E_q is the proper internal energy, given by (3.12). Here the difference from the standard Cardy-Verlinde formula (1.4) is the emergence of the proper internal energy E_q, which must be subtracted from the total energy. This result is reasonable because the proper internal energy (zero temperature energy) does not make its contribution to the entropy of CFT. In fact, following [2], we can also "derive" the formula (3.16) after considering there is an additional zero temperature energy in a certain thermodynamic system. The formula (3.16) is encouraging and the observation (3.14) is also interesting. To see whether it is universal, in the following subsections we will check the formula (3.16) with the charged black holes in the maximally supersymmetric gauged supergravities, in which some scalar fields are present.

3.2 Charged black holes in D=5 gauged supergravity

In this subsection we discuss the case of black holes in D=5, N=8 gauged supergravity. This solution has been found in [20] (also see [21]) as a special case (STU model) in the D=5, N=2 gauged supergravity.

The black hole solution has the metric

$$ds^2 = -(H_1H_2H_3)^{-2/3}fdt^2 + (H_1H_2H_3)^{1/3}(f^{-1}dr^2 + r^2d\Omega_3^2), \tag{3.17}$$

where

$$f = 1 - \frac{\mu}{r^2} + r^2 l^{-2} H_1 H_2 H_3, \quad H_i = 1 + \frac{q_i}{r^2}, \quad i = 1, 2, 3. \tag{3.18}$$

There are three real scalar fields X^i and three gauge potentials A^i

$$X^i = H_i^{-1}(H_1 H_2 H_3)^{1/3}, \quad A_t^i = \frac{\tilde{q}_i}{r^2 + q_i}, \quad i = 1, 2, 3. \tag{3.19}$$

Here the charges q_i are related to the physical electric charges $\tilde{q}_i$ via

$$q_i = \mu \sinh^2 \beta_i, \quad \tilde{q}_i = \mu \sinh \beta_i \cosh \beta_i. \tag{3.20}$$

The solution (3.17) has black hole horizon r_+ obeying the following equation

$$\mu = r_+^2 \left(1 + \frac{1}{l^2 r_+^4} \prod_{i=1}^{3} \rho_i \right), \quad \rho_i = r_+^2 + q_i. \tag{3.21}$$

The mass, Hawking temperature and entropy of black holes are

$$\begin{aligned} M &= \frac{\pi}{4G} \left(\frac{3}{2}\mu + \sum_i q_i \right), \\ T_{\mathrm{HK}} &= \frac{r_+^2}{2\pi \sqrt{\prod_i \rho_i}} \left[1 - \frac{\prod_i \rho_i}{l^2 r_+^4} \left(1 - r_+^2 \sum_i \frac{1}{\rho_i} \right) \right], \\ S &= \frac{\pi^2}{2G} \sqrt{\prod_i \rho_i}. \end{aligned} \tag{3.22}$$

The associated chemical potentials with the electric charges $\tilde{q}_i$ are

$$\phi_i = \frac{\pi}{4G} \frac{\tilde{q}_i}{\rho_i}. \tag{3.23}$$

As required, the first law of black hole thermodynamics is satisfied:

$$dM = T_{\mathrm{HK}} dS + \sum_i \phi_i d\tilde{q}_i. \tag{3.24}$$

Rescaling the boundary metric so that the three-dimensional sphere has the radius R, and using the relations (3.13) and (3.14) we obtain the Casimir energy

$$E_c = \frac{\pi l}{4GR} (\sum_i \rho_i - \sum_i q_i) = \frac{3\pi l r_+^2}{4GR}. \tag{3.25}$$

With this Casimir energy, the entropy of corresponding CFT can be written in the following form

$$S = \frac{2\pi R}{3} \sqrt{E_c(2(E - E_q) - E_c)}, \tag{3.26}$$

where the proper internal energy E_q and the thermal excitation energy are

$$E_q = \frac{\pi l}{4GR}\sum_i q_i, \quad E - E_q = \frac{3\pi l}{8GR}\left(r_+^2 + \frac{1}{l^2 r_+^2}\prod_i \rho_i\right). \tag{3.27}$$

Clearly, the expression (3.26) is a special case of (3.16) when $n = 3$, although the thermodynamics of the black hole solutions (3.17) is different from that of D=5 AdS Reissner-Nordström black holes because of the presence of three real scalar fields. Of course, the former degenerates to the latter when three charges are equal to each other. This can be seen from the solution (3.17).

3.3 Charged black holes in D=4 gauged supergravity

The black hole solution in D=4, N=8 gauged supergravity has been found in [22]. The metric has the form

$$ds^2 = -(H_1H_2H_3H_4)^{-1/2} f dt^2 + (H_1H_2H_3H_4)^{1/2}(f^{-1}dr^2 + r^2 d\Omega_2^2), \tag{3.28}$$

where

$$f = 1 - \frac{\mu}{r} + l^{-2}r^2\prod_{i=1}^{4} H_i, \quad H_i = 1 + \frac{q_i}{r}, \quad i = 1,2,3,4. \tag{3.29}$$

The four electric potentials are

$$A_t^i = \frac{\tilde{q}_i}{r + q_i}, \quad i = 1,2,3,4. \tag{3.30}$$

The physical charges $\tilde{q}_i$ are related to the charges q_i as the form (3.20). For the black hole solution (3.28), one has the horizon r_+ which satisfies the following equation,

$$\mu = r_+\left(1 + \frac{1}{l^2 r_+^2}\prod_i \rho_i\right), \quad \rho_i = r_+ + q_i, \quad i = 1,2,3,4. \tag{3.31}$$

The thermodynamics associated with black hole horizon can be easily found. The mass, Hawking temperature and entropy are

$$\begin{aligned}
M &= \frac{1}{4G}(2\mu + \sum_i q_i), \\
T_{\rm HK} &= \frac{r_+}{4\pi\sqrt{\prod_i \rho_i}}\left(1 - \frac{\prod_i \rho_i}{l^2 r_+^2} + \frac{\prod_i \rho_i}{l^2 r_+}\sum_i \frac{1}{\rho_i}\right), \\
S &= \frac{\pi}{G}\sqrt{\prod_i \rho_i},
\end{aligned} \tag{3.32}$$

respectively. The chemical potentials ϕ_i conjugating to the charges $\tilde{q}_i$ are

$$\phi_i = \frac{1}{4G}\frac{\tilde{q}_i}{\rho_i}, \quad i = 1, 2, 3, 4. \tag{3.33}$$

Once again, rescaling the boundary metric so that the two-dimensional sphere has the radius R, and repeating the calculations as the previous subsection, one has the Casimir energy

$$E_c = \frac{l}{4GR}(\sum_i \rho_i - \sum_i q_i) = \frac{lr_+}{GR}. \tag{3.34}$$

And the entropy can be rewritten as

$$S = \frac{2\pi R}{2}\sqrt{E_c(2(E - E_q) - E_c)}, \tag{3.35}$$

where the proper internal energy and the thermal excitation energy are

$$E_q = \frac{l}{4GR}\sum_i q_i, \quad E - E_q = \frac{l}{2GR}\left(r_+ + \frac{1}{l^2 r_+}\prod_i \rho_i\right). \tag{3.36}$$

The expression of entropy is the case of D=4 AdS Reissner-Nordström black holes. Although the thermodynamics of the solution (3.28) is also different from the one of D=4 AdS Reissner-Nordström black holes, the entropy of corresponding CFT's falls into the Cardy-Verlinde formula, which indicates the universality of the Cardy-Verlinde formula.

3.4 Charged black holes in D=7 gauged supergravity

The black hole solution in the D=7, N=4 gauged supergravity has the form [23, 21]

$$ds^2 = -(H_1 H_2)^{-4/5} f dt^2 + (H_1 H_2)^{1/5}(f^{-1}dr^2 + r^2 d\Omega_5^2), \tag{3.37}$$

where

$$f(r) = 1 - \frac{\mu}{r^4} + r^2 l^{-2} H_1 H_2, \quad H_i = 1 + \frac{q_i}{r^4}, \quad i = 1, 2. \tag{3.38}$$

The two gauge potentials in the solution (3.37) are

$$A_t^i = \frac{\tilde{q}_i}{r^4 + q_i}, \quad i = 1, 2. \tag{3.39}$$

As the case of D=5 or D=4, the physical charges $\tilde{q}_i$ are also related to the charges q_i via the relation (3.20). A standard calculation gives the thermodynamics of black hole

solution (3.37):

$$
\begin{aligned}
M &= \frac{\pi^2}{4G}\left(\frac{5}{4}\mu + \sum_i q_i\right), \\
T_{\rm HK} &= \frac{r_+^3}{\pi\sqrt{\prod_i \rho_i}}\left(1 - \frac{\prod_i \rho_i}{2r_+^6 l^2} + \frac{\prod_i \rho_i}{r_+^2 l^2}\sum_i \frac{1}{\rho_i}\right), \\
S &= \frac{\pi^3 r_+}{4G}\sqrt{\prod_i \rho_i}, \\
\phi_i &= \frac{\pi^2}{4G}\frac{\tilde{q}_i}{\rho_i},
\end{aligned}
\tag{3.40}
$$

where the constant μ is related to the black hole horizon r_+ as

$$
\mu = r_+^4 + \frac{1}{r_+^2 l^2}\prod_i \rho_i, \quad \rho_i = r_+^4 + q_i, \quad i = 1, 2. \tag{3.41}
$$

The first law here $dM = T_{\rm HK} dS + \sum_i \phi_i d\tilde{q}_i$ can be easily checked.

With the relations (3.13) and (3.14), we find the Casimir energy in this case is

$$
E_c = \frac{5\pi^2 l r_+^4}{8GR}, \tag{3.42}
$$

and the entropy of corresponding CFT has the form

$$
S = \frac{2\pi R}{5}\sqrt{E_c(2(E - E_q) - E_c)}, \tag{3.43}
$$

where the proper internal energy and the thermal excitation energy are

$$
E_q = \frac{\pi^2 l}{4GR}\sum_i q_i, \quad E - E_q = \frac{5\pi^2 l}{16GR}\left(r_+^4 + \frac{1}{r_+^2 l^2}\prod_i \rho_i\right). \tag{3.44}
$$

Once again, the entropy of corresponding CFT's to the charged black holes in D=7 gauged supergravity has the form of Cardy-Verlinde formula. Note that the solution (3.37) does not go to the one for a D=7 AdS Reissner-Nordström black hole even when two charges are equal, $q_1 = q_2$. This example further manifests that the Cardy-Verlinde formula (3.16) and the observation (3.14) on the pressure are universally valid for charged AdS black holes.

4 AdS black holes in higher derivative gravity

In this section we consider the AdS black holes in a special class of Lovelock gravity, which may be regarded as the most general generalization to higher dimensions of Einstein

gravity. The Lovelock action is a sum of the dimensionally continued Euler charactertics of all dimensions below the spacetime dimension (D) under consideration. The Lovelock action has an advantage that the resulting equations of motion contain no more than second derivatives of metric, as the pure Einstein-Hilbert action, but it includes $[D/2]$ arbitrary coefficients, which make it difficult to extract physical information from the solutions of equations of motion. It is possible to reduce those coefficients to two ones: a cosmological constant and a gravitational constant. By embedding the Lorentz group $SO(D-1,1)$ into a larger group, the anti-de Sitter group $SO(D-1,2)$, the Lovelock theory is divided into two different branches according to the spacetime dimension: odd dimensions and even dimensions. In odd dimensions, the action is the Chern-Simons form for the anti-de Sitter group; in even dimensions, it is the Euler density constructed with the Lorentz part of the anti-de Sitter curvature tensor. For details see [24].

The metric of a ($D = n+2$)-dimensional AdS black holes in the dimensionally continued gravity theory is [18, 24]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\sigma_n^2, \tag{4.1}$$

where

$$f(r) = \begin{cases} k - \left(\frac{2M}{r}\right)^{\frac{1}{m-1}} + \frac{r^2}{l^2} & \text{for} \quad D = 2m, \\ k - M^{\frac{1}{m-1}} + \frac{r^2}{l^2} & \text{for} \quad D = 2m-1, \end{cases} \tag{4.2}$$

where M is an integration constant and can be explained as the mass of black holes, in this case, it is implied that the anti-de Sitter space is viewed as the ground state of black holes [18]. $d\sigma_n^2 = \gamma_{ij}(x)dx^i dx^j$ denotes the line element of a n-dimensional hypersurface with constant curvature $n(n-1)k$. Without loss of generality, one may take $k = 1$, 0 and -1, respectively. When $k = 1$ the hypersurface σ_n is a positive constant curvature space, a simple case is just n-dimensional unit sphere S^n, as discussed in the above. When $k = -1$, the hypersurface is a negative constant curvature space. In this case, one can construct a closed hypersurface with arbitrary high genus via appropriate identification. When $k = 0$, the hypersurface is a zero curvature space, because of the reason explained in Section 2, we will not discuss this case.

In the solution (4.1), the horizon r_+ is determined by the equation

$$M = \begin{cases} \frac{r_+}{2}\left(k + \frac{r_+^2}{l^2}\right)^{m-1} & \text{for} \quad D = 2m, \\ \left(k + \frac{r_+^2}{l^2}\right)^{m-1} & \text{for} \quad D = 2m-1. \end{cases} \tag{4.3}$$

The Hawking temperature of black holes can be easily calculated, which is

$$T_{\rm HK} = \begin{cases} \frac{1}{4\pi(m-1)r_+}\left(k+\frac{(2m-1)r_+^2}{l^2}\right) & \text{for} \quad D=2m, \\ \frac{r_+}{2\pi l^2} & \text{for} \quad D=2m-1. \end{cases} \tag{4.4}$$

For the black holes in higher derivative gravity theories, the entropy is not simply one quarter of horizon area. In [18] we have presented a method to obtain the entropy of black holes in higher derivative gravity theories. That method is based on the fact that all black holes must obey the first law of thermodynamics $dM = T_{\rm HK} dS + \cdots$. Integrating the first law, we have

$$S = \int T_{\rm HK}^{-1} dM = \int_0^{r_+} T_{\rm HK}^{-1}\left(\frac{\partial M}{\partial r_+}\right) dr_+, \tag{4.5}$$

where we have imposed the physical assumption that the entropy vanishes when the horizon of black holes shrinks to zero. Evidently the entropy gained in this way is independent of the choice of ground state of black holes. With this formula, we can obtain easily the entropy of black holes in (4.1)

$$S = \begin{cases} \pi l^2 \left(\left(k+\frac{r_+^2}{l^2}\right)^{m-1} - k\right) & \text{for} \quad D=2m, \\ 4\pi(n-1)l\sum_{i=0}^{m-2}\left(\begin{array}{c} m-2 \\ i \end{array}\right)\frac{1}{2i+1}\left(\frac{r_+}{l}\right)^{2i+1} k^{m-2-i} & \text{for} \quad D=2m-1. \end{cases} \tag{4.6}$$

Now we are ready to check the Cardy-Verlinde formula with the AdS black holes in higher derivative gravity. Rescaling the metric so that the constant curvature hypersurface has the radius R, we then have the energy E and temperature T of the corresponding CFT's

$$E = \frac{l}{R} M, \quad T = \frac{l}{R} T_{\rm HK}, \tag{4.7}$$

and the entropy S of the CFT's is still given by the entropy (4.6) of black holes.

In the case for the even dimensional black holes, namely, $D = 2m$, we find the Casimir energy is

$$E_c = \frac{kr_+ l}{2R}\left[2m-1-\frac{l^2}{r_+^2}\left(\left(k+\frac{r_+^2}{l^2}\right)^{m-1}-k\right)\right], \tag{4.8}$$

and furthermore we have

$$2E - E_c = \frac{r_+ l}{2R}\left[\left(2-\frac{l^2}{r_+^2}k\right)\left(k+\frac{r_+^2}{l^2}\right)^{m-1} - (2m+1)k + \frac{l^2}{r_+^2}k^2\right]. \tag{4.9}$$

So for both cases $k = \pm 1$, we cannot put the entropy in (4.6) into the form of Cardy-Verlinde formula.

In the case for odd dimensions ($D = 2m - 1$), the Casimir energy is

$$E_c = \frac{2(m-1)l}{R}\left(k + \frac{r_+^2}{l^2}\right)^{m-1} - \frac{2(m-1)(2m-3)r_+}{R}\sum_{i=0}^{m-2}\left(\begin{array}{c} m-2 \\ i \end{array}\right)\frac{1}{2i+1}\left(\frac{r_+}{l}\right)^{2i+1} k^{m-2-i}. \tag{4.10}$$

Again, we cannot put the entropy in (4.6) into the form of the Cardy-Verlinde formula. To clearly see this, let us consider a special dimension $D = 5$. In this case, the action of the gravity theory is the Einstein-Hilbert action plus a Gauss-Bonnet term. For such a black hole, the entropy is

$$S = 8\pi r_+ \left(k + \frac{r_+^2}{3l^2}\right). \tag{4.11}$$

And the Casimir energy for the corresponding CFT is found to be

$$E_c = \frac{4kl}{R}\left(k - \frac{r_+^2}{l^2}\right), \tag{4.12}$$

and the extensive energy is

$$2E - E_c = \frac{2l}{R}\left(\left(2k + \frac{r_+^2}{l^2}\right)^2 - 5k^2\right). \tag{4.13}$$

This special example clearly indicates that the entropy (4.11) does not fall into the form of Cardy-Verlinde formula (1.4).

5 Conclusions

The Cardy-Verlinde formula recently proposed by E. Verlinde [2], relates the entropy of a certain CFT to its total energy and Casimir energy in arbitrary dimensions. In the spirit of AdS/CFT correspondence, this formula has been shown to hold exactly for the cases of AdS Schwarzschild black holes and AdS Kerr black holes.

In this paper we have further checked the Cardy-Verlinde formula with some typical examples of black holes with AdS asymptotics. They are hyperbolic AdS black holes, AdS Reissner-Nordström black holes, charged black holes in D=5, D=4, and D=7 maximally supersymmetric gauged supergravities, and AdS black holes in higher derivative gravity. For the hyperbolic AdS black holes, the formula holds if we choose the "massless" black

hole as the ground state of black holes (otherwise, this formula will no longer hold), but in this case, the Casimir energy is found to be negative [see (2.10)]! Obviously, further investigations are needed for the hyperbolic AdS black holes. In fact, the understanding of the AdS/CFT correspondence is poor for the thermodynamics of the hyperbolic black holes so far [17].

For the AdS Reissner-Nordström black holes in arbitrary dimensions and charged black holes in D=5, D=4 and D=7 maximally supersymmetric gauged supergravities, the Cardy-Verlinde formula can also hold by subtracting the proper internal energy from the total energy [see (3.16)]. The proper internal energy corresponds to the contribution of supersymmetric backgrounds. In the thermodynamics of corresponding CFT's, we can view the proper internal energy as the zero temperature energy, which has the contribution to the free energy, but not to the entropy of thermodynamic system. Therefore our result (3.16) is reasonable and can be viewed as an extension of Cardy-Verlinde formula (1.4). In addition, it might be worth mentioning that for the corresponding CFT to the charged AdS black holes, its pressure is given by (3.14), namely $p = (E - E_q)/nV$. The quantity $E - E_q$ has an interpretation as the thermal excitation energy of CFT's.

We have also considered the AdS black holes in the Lovelock gravity and found that the entropy of corresponding CFT's cannot be put into the Cardy-Verlinde formula. This seems reasonable since the Cardy-Verlinde formula (1.3) was derived through the assumption that the first subleading correction to the extensive part of the energy scales like $E_c(\lambda S, \lambda V) = \lambda^{1-2/n} E_c(S, V)$. The corrections from higher derivative terms are beyond the scope of the scaling.

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References

[1] J.L. Cardy, Nucl. Phys. **B270**, 317 (1986).

[2] E. Verlinde, hep-th/0008140.

[3] J.D. Bekenstein, Phys. Rev. D **23**, 287 (1981).

[4] R. Bousso, hep-th/0012052.

[5] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998), hep-th/9711200.

[6] S. Gubser, I. Klebanov and A. Ployakov, Phys. Lett. **B428**, 105 (1998), hep-th/9802109.

[7] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998), hep-th/9802150.

[8] E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998), hep-th/9803131.

[9] D. Klemm, A.C. Petkou and G. Siopsis, hep-th/0101076.

[10] D. Kutasov and F. Larsen, JHEP **0001**, 001 (2001), hep-th/0009244.

[11] F.-L. Lin, hep-th/0010127.

[12] S. Nojiri and S. Odintsov, hep-th/0011115.

[13] B. Wang, E. Abdalla and R.K. Su, hep-th/0101073.

[14] R. Brustein, S. Foffa and G. Veneziano, hep-th/0101083.

[15] I. Savonije and E. Verlinde, hep-th/0102042.

[16] D. Birmingham, Class. Quant. Grav. **16**, 1197 (1999), hep-th/9808032.

[17] R. Emparan, JHEP **9906**, 036 (1999), hep-th/9906040.

[18] R.G. Cai and K.S. Soh, Phys. Rev. D **59**, 044013 (1999), gr-qc/9808067.

[19] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, Phys. Rev. D **60**, 064018 (1999), hep-th/9902170.

[20] K. Behrndt, M. Cvetic and W.A. Sabra, Nucl. Phys. **B553**, 317 (1999), hep-th/9810227.

[21] M. Cvetic and S. Gubser, JHEP **9904**, 024 (1999), hep-th/9902195.

[22] M.J. Duff and J.T. Liu, Nucl.Phys. **B554**, 237 (1999), hep-th/9901149.

[23] J.T. Liu and R. Minasian, Phys. Lett. **B457**, 39 (1999), hep-th/9903269.

[24] M. Banados, C. Teitelboim and Z. Janelli, Phys. Rev. D **49**, 975 (1994).